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Hybrid Orthonormal Bernstein and Block-Pulse functions wavelet scheme for solving the 2D Bratu problem

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ABSTRACT

In this paper, an efficient numerical scheme is settled for solving two-dimensional Bratu–Gelfand problem, namely Hybrid Orthonormal Bernstein and Block-Pulse functions wavelet (HOBW) is presented for boundary value problems administered by nonlinear partial differential equations which effectively combines the Orthonormal Bernstein, Block-Pulse functions and the generalized wavelet. Operational Matrix of integration is utilized to provide an approximate result of the BG problems. By using the Operational Matrix, differentiation is changed to the nonlinear system of equations which can be disbanded via the Newton Raphson technique. As per our concentrated inquiry there is no exact solution of the problem and can solve the problem with higher accuracy than the methodologies used to solve this problem. The result is plotted for different values of λ then compared with the previous numerical results obtained.

Introduction

The Bratu–Gelfand (BG) problem is an elliptic nonlinear partial differential equation. It is performing from the improvement of torching model in fuel combustion theory [1–3]. BG has the variety of application in science and chemistry to depict chemical and physical models like; fuel ignition model of the thermal combustion theory, nanotechnology models, radiative heat transfer chemical reaction processes and chemical reactor theory and nanotechnology [4]. Numerical and analytical techniques are applied for 1D and 2D BG equation [5–14]. This problem is contemplated by numerous creators utilizing different techniques such as the wavelet homotopy analysis method (WHAM) [15], the finite difference [16], Spline method [17], VIM and Green's function [18–20], The relation between the Legendre and wavelet methods [21,22], differential transformation [23], Taylor's decomposition method [24], Lie-group shooting method [25], Sinc-Galerkin method [26] and Block Nyström method [27]. Most of these examinations are computationally expensive. Further, they could not describe more than one solution for such nonlinear issue.

In recent papers, Orthonormal Bernstein and Block-Pulse functions have become increasingly famous in the numerical solution of integral and partial differential equation. The benefits of this strategy lie in its simple utilize and adaptability. Also, HOBW method can yield accurate results with relatively much fewer points compared with the previous techniques such as the finite difference and finite element methods. These results confirm that, HOBW is very effective and more accurate than some other methods.

The outline of the paper is organized as follows. In Section "Dominant equation" we introduce a formulation of two-dimensional BG equation. In Section "The HOBW scheme and their Properties", the analysis of the method has been discussed. Description of the method has been discussed in Section "Description of the method". Finally, we report our numerical results and demonstrate the efficiency and accuracy of the proposed numerical scheme in Section "Illustrative examples".

Dominant equation

Two-dimensional BG equation is a nonlinear boundary value problem that can be written as [28]:

$$\frac{\partial^2}{\partial x^2}u(x, y) + \frac{\partial^2}{\partial y^2}u(x, y) + \lambda e^{u(x,y)} = 0, x, y \in \Omega \tag{2.1}$$

Where Ω: $\{(x, y) \in 0 \le x \le 1, 0 \le y \le 1\}$, and λ denotes the reaction term, $\lambda > 0$. The confines conditions are defined as;

$$u(0, y) = u(1, y) = u(x, 0) = u(x, 1) = 0, x, y \in \partial\Omega \tag{2.2}$$

where $\partial\Omega$ means the unit square confines $\{x, y \in \partial\Omega = [0; 1] * [0; 1]\}$.

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However, the two-dimensional analytical result that satisfies the confines conditions and additionally the differential equation at just one collocation point (0.5, 0.5) was presented in [27]:

$$u(x,y) = 2\ln\left[\frac{\left[\cosh\left(\frac{\theta}{4}\right)\cosh\left[\left(x-\frac{1}{2}\right)\left(y-\frac{1}{2}\right)\theta\right]\right]}{\left[\cosh\left(\left(x-\frac{1}{2}\right)\frac{\theta}{2}\right)\cosh\left(\left(y-\frac{1}{2}\right)\frac{\theta}{2}\right)\right]}\right] \tag{2.3}$$

Where, $\theta^2 = \lambda\left(\cosh\frac{\theta}{4}\right)^2$, $\theta^2 = \frac{\lambda_c}{4}\sinh\left(\frac{\theta}{4}\right)\cosh\left(\frac{\theta}{4}\right)$ and $\lambda = \frac{\theta^2}{2}\left(\mathrm{sech}\frac{\theta}{4}\right)^2$.

The HOBW scheme and their properties

One dimensional HOBW method

Wavelets constitute a group of functions constructed from dilation and translation of a single function $\Psi(x)$ called the mother wavelet. In which parameter of dilation a and parameter of translation b vary continuously.

$$\Psi_{a,b}(x) = |a|^{\frac{-1}{2}}\Psi\left(\frac{t-b}{a}\right),\ a, b \in R,\ a \neq 0.$$

By letting $a$and b be discrete values such as $a = {a_0}^{-k}$, $b = nb_0{a_0}^{-k}$, $a_0 > 1$, $b_0 > 0$, n and k are positive integers, we attain the family of discrete wavelets:

$$\Psi_{k,n}(x) = |a_0|^{\frac{k}{2}}\Psi({a_0}^k t - nb_0),\ n, k \in Z^+ \tag{3.1}$$

Then we see that $\Psi_{k,n}(x)$ forms a wavelet basis for $L^2(R)$.In detail, when, $a_0 = 2$, $b_0 = 1$,then $\Psi_{k,n}(x)$ forms basis. Here, $HOBW_{i,j}(t) = HOBW(k, i, j, t)$ involves four arguments, $i = 1, \cdots, 2^{k-1}$, k is to be any positive integer, j is the degree of the Bernstein polynomials, and$t$ is the normalized time. They $HOBW_{i,j}(t)$ are defined on $[0, 1)$ as [29]:

$$HOBW_{i,j}(x) = \begin{cases} 2^{\frac{k-1}{2}}\binom{l}{j}(2^{k-1}x - r_{i-1})^j(r_i - 2^{k-1}x)^{l-j} & r_{i-1} \leqslant x \leqslant r_i \\ 0 & otherewise \end{cases} \tag{3.2}$$

$$r_i = \frac{(2^{k-1}-i)a + ib}{2^{k-1}}$$

Where $i = 1, 2, \cdots, 2^{k-1}$, $j = 0, 1, \cdots, M-1$ and k is a positive integer. Thus, we attain our new basis as $\{HOBW_{1,0}, HOBW_{1,1}, \cdots, HOBW_{2^{k-1},M-1}\}$ and any function is truncated with them.

The $HOBW$ detect orthonormal basis is:

$$\begin{aligned} HOBW = [&HOBW_{1,0,1,0},\ldots, HOBW_{1,0,1,M'-1}, HOBW_{1,0,2,0},\ldots,HOBW_{1,0,2,M'-1},\ldots,HOBW_{1,0,2^{k'-1}},\ldots,\\ &HOBW_{1,0,\ 2^{k'-1},M'-1},\ldots, HOBW_{1,M-1,1,0},\ldots,HOBW_{n,m,n'm'},\ldots,HOBW_{1,M-1,2,0},\ldots,HOBW_{1,M-1,2,M'-1},\\ &HOBW_{1,M-1,2^{k'-1},\ 0},\ldots, HOBW_{1,M-1,2^{k'-1},\ M'-1},\ldots, HOBW_{2,0,1,\ M'-1},\ldots, HOBW_{2,0,2,0},\ldots,\\ &HOBW_{2,0,2,\ M'-1},\ldots, HOBW_{2,0,2^{k'-1},0},\ldots, HOBW_{2,0,2^{k'-1},M'-1},\ldots, HOBW_{2,M-1,1,0},\ldots, HOBW_{2,M-1,1,M'-1}\\ &HOBW_{2,M-1,2,0},\ldots, HOBW_{2,M-1,2,\ M'-1},\ldots, HOBW_{2,M-1,2^{k'-1},\ 0},\ldots, HOBW_{2,M-1,2^{k'-1},\ M'-1},\ldots,\\ &HOBW_{2^{k-1},0,\ 1,0},\ldots, HOBW_{2^{k-1},0,\ 1,M'-1}, HOBW_{2^{k-1},0,2,0},\ldots,HOBW_{2^{k-1},0,\ 2,M-1},\ldots,\\ &HOBW_{2^{k-1},0,2^{k'-1},0'},\ldots,HOBW_{2^{k-1},0,2^{k'-1},M'-1},\ldots,HOBW_{2^{k-1},M-1,2^{k'-1},0},\ldots,HOBW_{2^{k-1},M-1,2^{k'-1},M'-1}]^T \end{aligned} \tag{3.9}$$

$$(HOBW_{i,j}(x), HOBW_{i',j'}(x)) = \begin{cases} 1 (i,j) = (i',j') \\ 0 (i,j) \neq (i',j') \end{cases}$$

where (.,.) called the inner product in $L^2[0, 1)$ The $HOBW$has compact support $\left[\frac{i-1}{2^{k-1}}, \frac{i}{2^{k-1}}\right]$.

Two dimensional HOBW scheme

Two-dimensional $HOBW$ can be expressed as the product of one-dimensional $HOBW$ as follows:

$$HOBW_{i,j,n,m}(x,t) = \begin{cases} HOBW_{i,j}(x)\ HOBW_{n,m}(t) & r'_{i-1} \leqslant t \leqslant r'_i \\ 0 & otherewise \end{cases} \tag{3.3}$$

where,

$$HOBW_{i,j}(x) = \begin{cases} 2^{\frac{k-1}{2}}\binom{l}{j}(2^{k-1}x - r_{i-1})^j(r_i - 2^{k-1}x)^{l-j} & r_{i-1} \leqslant x \leqslant r_i \\ 0 & otherewise \end{cases} \tag{3.4}$$

$$HOBW_{n,m}(t) = \begin{cases} 2^{\frac{k'-1}{2}}\binom{n'}{m}(2^{k'-1}t - r'_{i-1})^m(r'_i - 2^{k'-1}t)^{n'-m} & r'_{i-1} \leqslant t \leqslant r'_i \\ 0 & otherewise \end{cases} \tag{3.5}$$

$$r'_i = \frac{(2^{k'-1}-i)a + ib}{2^{k-1}}$$

where $i = 1, 2, \cdots, 2^{k-1}$, $n = 1, 2, \cdots, 2^{k'-1}$, $j = 0, 1, \cdots, M-1$, $m = 0, 1, \cdots, M'-1$ and k is a positive integer.

Function approximation by using the HOBW functions

Any function $u(x, t)$, which is integrable is truncated by using the HOBW scheme as follows:

$$u(x,t) = \sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty} u_{i,j,n,m}HOBW_{i,j,n,m}(x,t), \tag{3.6}$$

the infinite series truncated to the following:

$$u(x,t) = \sum_{i=1}^{2^{k-1}}\sum_{j=0}^{M-1}\sum_{n=1}^{2^{k'-1}}\sum_{m=0}^{M'-1} u_{i,j,n,m}HOBW_{i,j,n,m}(x,t) = U^T HOBW(x,t) \tag{3.7}$$

where U^T and $HOBW$ are $2^{k-1}2^{k-1}MM' \times 1$ vector defined as follows:

$$\begin{aligned} U = [&U_{1,0,1,0},\ldots, U_{1,0,1,M'-1}, U_{1,0,2,0},\ldots,U_{1,0,2,M'-1},\ldots,U_{1,0,2^{k'-1}},\ldots,\\ &U_{1,0,\ 2^{k'-1},M'-1},\ldots, U_{1,M-1,1,0},\ldots,U_{n,m,n'm'},\ldots,U_{1,M-1,2,0},\ldots,U_{1,M-1,2,M'-1},\\ &U_{1,M-1,2^{k'-1},\ 0},\ldots, U_{1,M-1,2^{k'-1},\ M'-1},\ldots, U_{2,0,1,\ M'-1},\ldots, U_{2,0,2,0},\ldots,\\ &U_{2,0,2,\ M'-1},\ldots, U_{2,0,2^{k'-1},0},\ldots, U_{2,0,2^{k'-1},M'-1},\ldots, U_{2,M-1,1,0},\ldots, U_{2,M-1,1,M'-1}\\ &U_{2,M-1,2,0},\ldots, U_{2,M-1,2,\ M'-1},\ldots, U_{2,M-1,2^{k'-1},\ 0},\ldots, U_{2,M-1,2^{k'-1},\ M'-1},\ldots,\\ &U_{2^{k-1},0,\ 1,0},\ldots, U_{2^{k-1},0,\ 1,M'-1}, U_{2^{k-1},0,2,0},\ldots,U_{2^{k-1},0,\ 2,M-1},\ldots,\\ &U_{2^{k-1},0,\ 2^{k'-1},0'},\ldots, U_{2^{k-1},0,\ 2^{k'-1},M'-1},\ldots\\ &,U_{2^{k-1},M-1,\ 2^{k'-1},0},\ldots,U_{2^{k-1},M-1,\ 2^{k'-1},M'-1}]^T \end{aligned} \tag{3.8}$$

HOBW operational matrix of integration

The OM instituted in $HOBW$ for variable x is obtained as follows:

$$\int_0^x HOBW(\delta,t)d\delta = \int_0^x (HOBW(\delta)\cdot HOBW(t))d\delta$$
$$= \left(\int_0^x (HOBW(\delta))d\delta\right)\cdot HOBW(t)$$
$$= PHOBW(\delta)\cdot IHOBW(t)$$
$$= (P\cdot I)(HOBW(x)\cdot HOBW(t)) = \widehat{A}HOBW(x,t)$$

The OM instituted in $HOBW$ for variable t is obtained as follows:

$$\int_0^t HOBW(x,\delta)d\delta = \int_0^t (HOBW(x)\cdot HOBW(\delta))d\delta$$
$$= HOBW(x)\cdot\left(\int_0^t (HOBW(\delta))d\delta\right)=(IHOBW(t)\cdot$$
$$(PHOBW(t)) = (I\cdot P)(HOBW(x)\cdot HOBW(t))$$
$$= \widehat{B}HOBW(x,t)$$

Thus, we have $\int_0^x (HOBW(\delta,t)d\delta = \widehat{A}HOBW(x,t)$

where, $\widehat{A} = P\cdot I$ is $MM^{'}2^{k-1}2^{k^{'}-1}$. $MM^{'}2^{k-1}2^{k^{'}-1}$ matrix in which P is a matrix of dimension $2^{k-1}M$.

Description of the method

For the numerical result of Equation

$$\frac{\partial^2}{\partial x^2}u(x,y) + \frac{\partial^2}{\partial y^2}u(x,y) + \lambda e^{u(x,y)} = 0,\ x,y\in\Omega \tag{4.1}$$

we start with the approximating of $\frac{\partial^2}{\partial x^2}u(x,y)$

$$\frac{\partial^2}{\partial x^2}u(x,y) = \sum_{i=1}^{2^{k^{'}-1}}\sum_{j=0}^{2M}\sum_{n=1}^{2^{k^{'}-1}}\sum_{m=0}^{2M} C_{i,j}HOBW_{i,j,n,m}(x,t) = C^T HOBW(x,t) \tag{4.2}$$

Integrating Eq. (4.2) regarding to x, gives

$$\frac{\partial u(x,t)}{\partial x} = \frac{\partial u(0,t)}{\partial x} + C^T\widehat{A}HOBW(x,t) \tag{4.3}$$

Again, integrating Eq. (4.3) regarding to x, yields

$$u(x,t) = x\frac{\partial u(0,t)}{\partial x} + C^T\widehat{A}\widehat{A}HOBW(x,t) \tag{4.4}$$

The hire of second confines condition $u(1,t)=0$ and Eq. (4.4) yields

$$u(1,t) = x\frac{\partial u(0,t)}{\partial x} + C^T\widehat{A}\widehat{A}HOBW(1,t) = 0 \tag{4.5}$$

or

$$\frac{\partial u(0,t)}{\partial x} = -C^T\widehat{A}\widehat{A}HOBW(1,t) \tag{4.6}$$

By inserting Eq. (4.6) into the Eq. (4.3), one can obtain,

$$\frac{\partial u(x,t)}{\partial x} = -C^T\widehat{A}\widehat{A}HOBW(1,t) + C^T\widehat{A}HOBW(x,t)$$
$$= -C^T\widehat{A}\widehat{A}K^T HOBW(x,t) + C^T\widehat{A}HOBW(x,t) \tag{4.7}$$

Therefore

$$\frac{\partial u(x,t)}{\partial x} = -C^T\widehat{A}(-\widehat{A}K^T + I)HOBW(x,t) \tag{4.8}$$

$$u(x,t) = -C^T\widehat{A}\widehat{A}xHOBW(1,t) + C^T\widehat{A}\widehat{A}HOBW(x,t)$$
$$= -C^T\widehat{A}\widehat{A}KK^T HOBW(x,t) + C^T\widehat{A}\widehat{A}HOBW(x,t)u(x,t)$$
$$= -C^T\widehat{A}\widehat{A}(I-KK^T)HOBW(x,t) \tag{4.9}$$

By Similarly

$$\frac{\partial^2}{\partial y^2}u(x,y) = \sum_{i}^{2^{k^{'}-1}}\sum_{j}^{2M}\sum_{n}^{2^{k^{'}-1}}\sum_{m}^{2M} C_{i,j}HOBW_{i,j,n,m}(x,t) = C^T\widehat{B}HOBW(x,t) \tag{4.10}$$

$$\frac{\partial^2}{\partial x^2}u(x,y) + \frac{\partial^2}{\partial y^2}u(x,y) + \lambda e^{u(x,y)} = 0 \tag{4.11}$$

On suitable substitution from Eqs. (4.2) to (4.11) into (4.1), BG problem can be reduced to the following iterative one:

$$-C^T\widehat{A}\widehat{A}(I-KK^T)HOBW(x,t) + lnC^T\widehat{B}\ HOBW(x,t)$$
$$+ C^T\widehat{B}HOBW(x,t) = 0 \tag{4.12}$$

We now collocate Eq. at $2^{k^{'}-1}M$ at x_i, t_i

$$C^T\widehat{A}\ HOBW(x_i,t_i) + C^T\widehat{B}\ HOBW(x_i,t_i)$$
$$+ F(x_i,t_i,-C^T\widehat{A}\widehat{A}(I-KK^T)\ HOBW(x_i,t_i)) = 0 \tag{4.13}$$

Suitable collocation points are the zeros of Chebyshev polynomial [30]

at

$$x_i = \cos\left(\frac{(2i+1)\pi}{2^k M}\right) \tag{4.14}$$

at

$$t_i = \cos\left(\frac{(2j+1)\pi}{2^{k^{'}}M^{'}}\right) \tag{4.15}$$

Eq. (4.13) is a system of $2^{k-1}2^{k^{'}-1}MM^{'}$ nonlinear equations which can be disbanded for the elements of C by the well-known Newton-Raphson method.

Illustrative examples

The execution of the approach method for BG problem is illustrated; some cases are presented. The per shapes reveal that the scheme is profitable and straightforward. All computations were carried out using Maple on a personal computer. Also, the graph of approximate solutions, at $2^{k-1}=4$, $2^{k^{'}-1}=4$, $M=3$ and $M^{'}=3$, for different values of $\lambda = 0.05, 0.5, 1, 4, 7.027661438$ have been plotted in Figs. 5.1–5.5. Results in Tables 5.1 and 5.2.

Error analysis

Theorem 1. *A function $u(x,t)\in L_w^2$ ([0,1]) with bounded second derivative, say $\left|\frac{\partial^2}{\partial x^2}u(x,y)\right| \le N$ can be expanded using HOBW and converges to $u(x,t)$, i.e.,*

$$u(x,t) = -C^T\widehat{A}\widehat{A}(I-KK^T)HOBW(x,t)$$

Since the truncated HOBW is a proximate outcome of BG problem, so the error function $E(x,t)$ for $u(x,t)$ as follows:

$$E(x,t) = |u(x,t) - (-C^T\widehat{A}\widehat{A}(I-KK^T)HOBW(x,t))|$$

The following theorem gives an error bound of the proximate the outcome by using HOBW method.

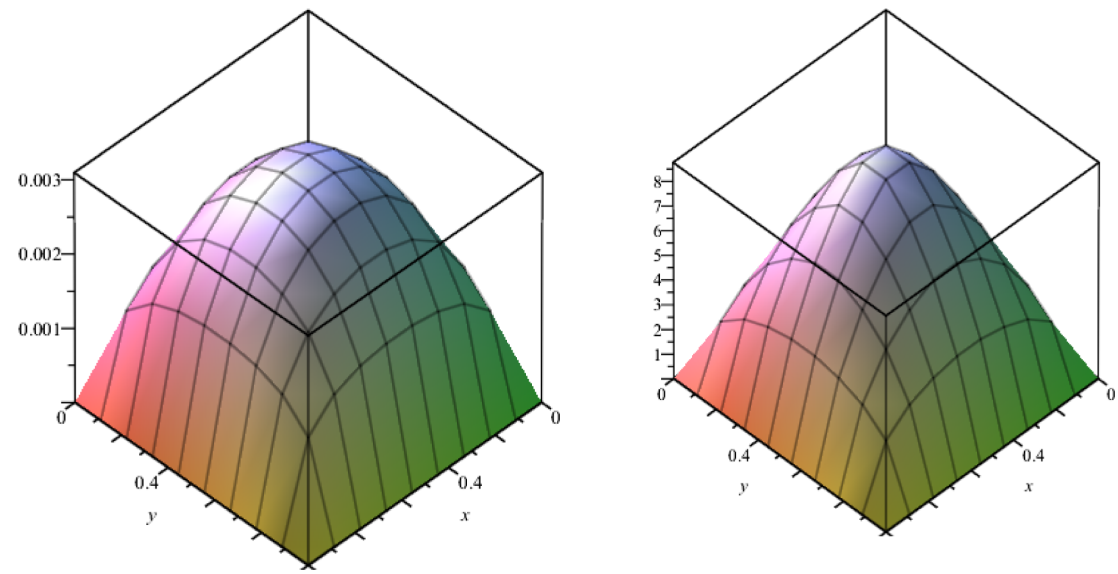

Fig. 5.1. The outcome of BG equation for $\lambda = 0.05$.

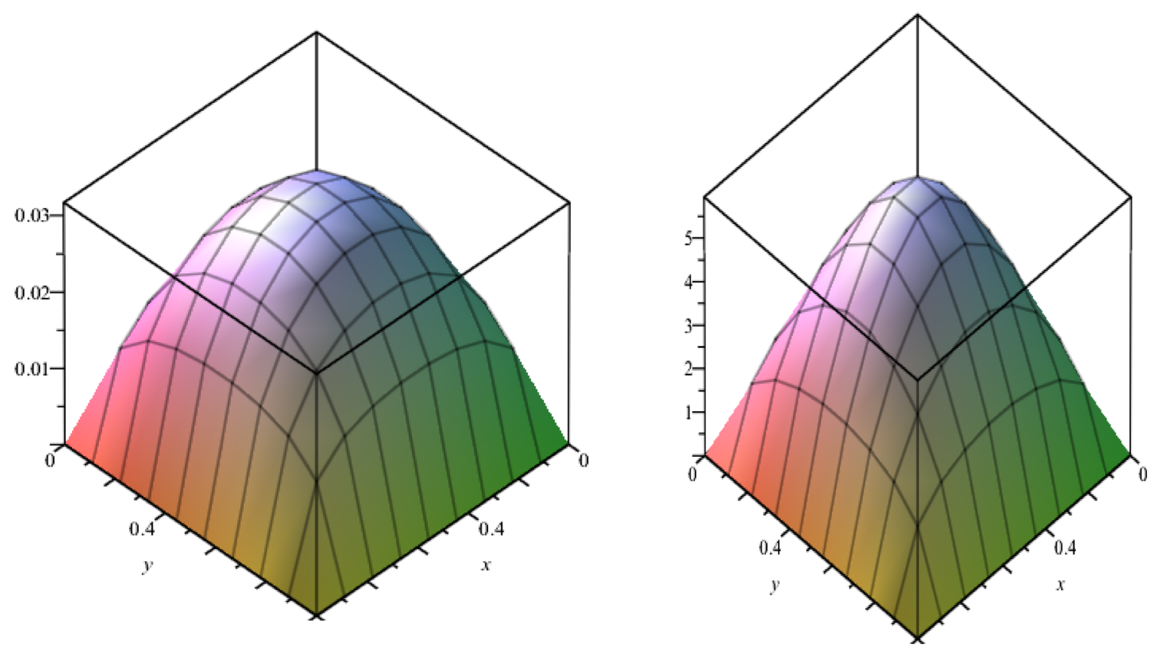

Fig. 5.2. The outcome of BG equation for $\lambda = 0.5$.

Theorem 2. *Suppose that $u(x, t) \in C^m[0, 1]$ and $-C^T\widehat{A}\widehat{A}(I - KK^T)HOBW(x, t)$ is the approximate the outcome using the HOBW. Then the error bound would be obtained as follows:*

$$E(x, t) \leqslant \left\| \frac{2}{(m!)^2\ 4^{2m}2^{2m(k-1)}} \max_{x\in[0,1]} |u^{2m}(x, t)| \right\|^2$$

Proof. Using the norm definition in the inner product space, we have:

$$\|E^2\| \leqslant \int_0^1\int_0^1 [u(x, t) - -C^T\widehat{A}\ \ \widehat{A}(I - KK^T\ \ )HOBW(x, t)]^2 dxdt$$

Because the interval [0, 1] is divided into 2^{k-1} subintervals I_n= then we can obtain

$$\|E^2\| = \int_0^1\int_0^1 u[(x, t) - -C^T\widehat{A}\ \ \widehat{A}(I - KK^T)HOBW(x, t)]^2\ dx\ \ dt$$

$$= \sum_{k=1}^{2^{k-1}}\sum_{k'=1}^{2^{k'-1}}\int_{\frac{i}{2^{k-1}}}^{\frac{i-1}{2^{k-1}}}\int_{\frac{n}{2^{k'-1}}}^{\frac{n-1}{2^{k'-1}}} [u(x, t) - -C^T\widehat{A}\widehat{A}(I - KK^T)HOBW(x, t)]^2 dt$$

$$= \sum_{k=1}^{2^{k-1}}\sum_{k'=1}^{2^{k'-1}}\int_{\frac{i}{2^{k-1}}}^{\frac{i-1}{2^{k-1}}}\int_{\frac{n}{2^{k'-1}}}^{\frac{n-1}{2^{k'-1}}} [u(x, t) - P_m(x, t)]^2 dt$$

$$\leqslant \sum_{k=1}^{2^{k-1}}\sum_{k'=1}^{2^{k'-1}}\int_{\frac{i}{2^{k-1}}}^{\frac{i-1}{2^{k-1}}}\int_{\frac{n}{2^{k'-1}}}^{\frac{n-1}{2^{k'-1}}} \left[\frac{2}{m!\ \ 4^{m}2^{m(k-1)}}\max_{x\in[0,1]} |u^{2m}(x, t)|\right]^2\ dx\ \ dt$$

$$= \int_0^1\int_0^1 \left[\frac{2}{m!\ \ 4^{m}2^{m(k-1)}}\max_{x\in[0,1]} |u^{2m}(x, t)|\right]^2\ dx\ \ dt$$

$$= \left\| \frac{2}{(m!)^2\ \ 4^{2m}2^{2m(k-1)}}\max_{x\in[0,1]} |u^{2m}(x, t)| \right\|^2$$

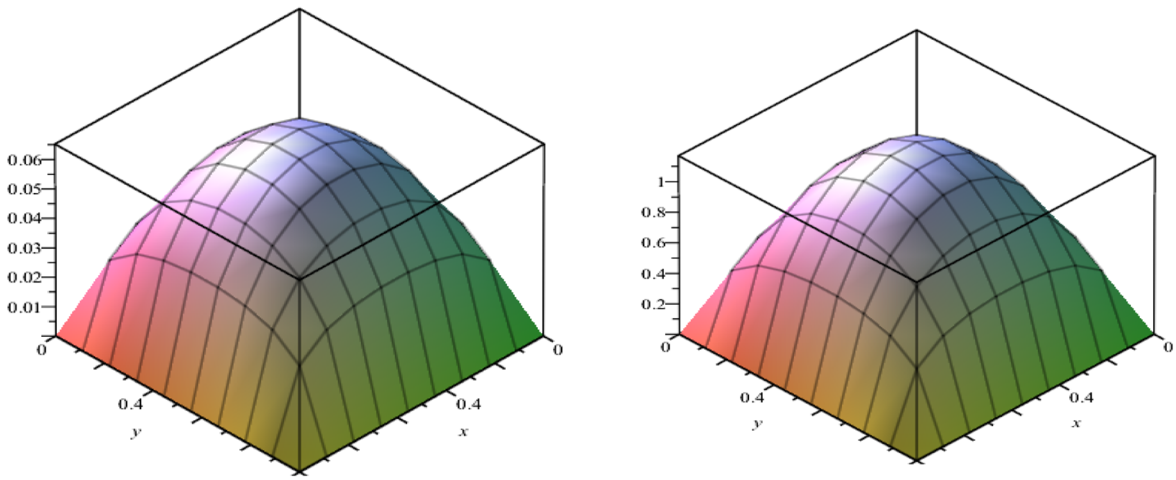

Fig. 5.3. The outcome of BG equation for $\lambda = 1$.

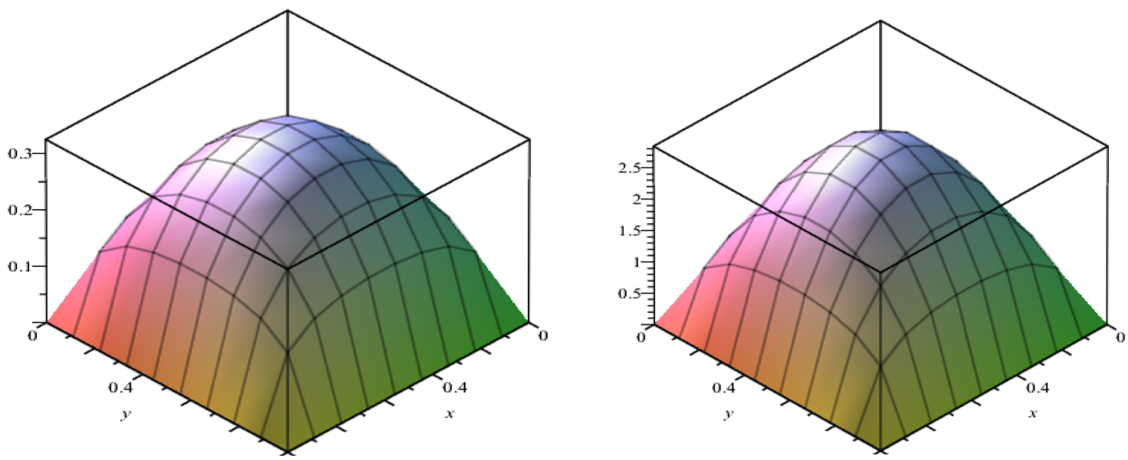

Fig. 5.4. The outcome of BG equation for $\lambda = 4$.

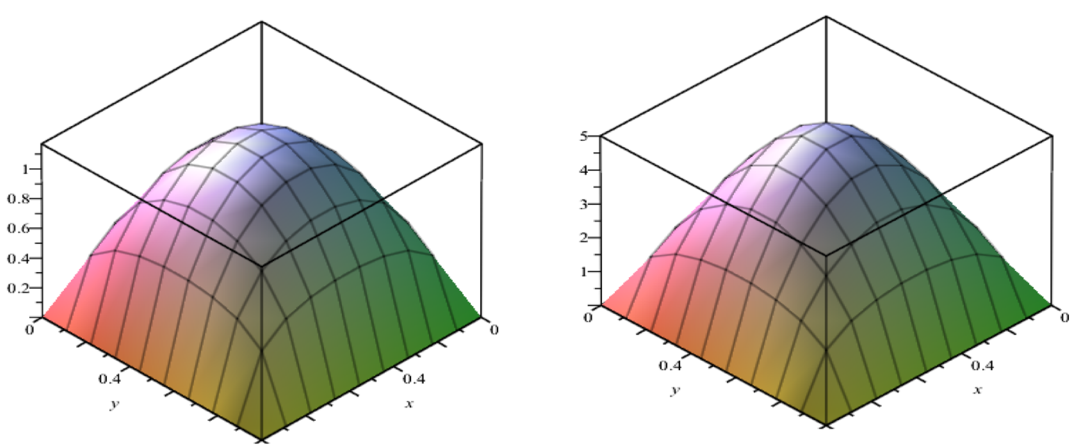

Fig.5.5. The outcome of BG equation for $\lambda = 7.027661438$.

Table 5.1

Comparison between the HOBW at $2^{k-1} = 4$, $2^{k'-1} = 4$, $M = 3$ and $M' = 3$ and [31] at $N = 7 \times 7$ for different λ.

λ	u_{lower} using HOBW	u_{upper} using HOBW	u_{lower} [31]	u_{upper} [31]
0.05	0.0031332	9.0779426	0.0031059	9.0777656
1	0.0660366	5.1357731	0.0660123	5.1352663

Conclusion

A numerical scheme for resolving the 2DBG equations is proposed. This scheme is instituted in HOBW. We introduce an approximation approach to settle the BG problem. The HOBW is much more reliable and achieved much more quickly. We hired the MAPLE algorithm fsolve to decide the nonlinear system. The results obtained here are compared with those obtained using the Iterative differential quadrature method

Table 5.2
Comparison between the HOBW at $2^{k-1}=4$, $2^{k'-1}=4$, $M=3$ and $M'=3$and [31] at $N=11\times 11$ for different λ.

λ	u_{lower} using HOBW	u_{upper} using HOBW	u_{lower} [31]	u_{upper} [31]
0.05	0.0031332	9.0779426	0.0031332	9.0779426
1	0.0660366	5.1357731	0.0660366	5.1357731

[31], the HOBW scheme provides accurate outcomes for both upper and lower branches for critical values of λ. The solution obtained using the HOBW method demonstrate that this approach can solve the 2DBG effectively. It was apparent that, the HOBW method for a specific estimation of k, M, as M', k' increased, the accuracy was increased, and for a certain value of M', k' as k, M increased, the accuracy was increased as well.

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Appendix A. Supplementary data

Supplementary data to this article can be found online at https://doi.org/10.1016/j.rinp.2018.11.056.

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